NORTH SYDNEY GIRLS HIGH SCHOOL

Year 12

HSC Mathematics Extension 1 Assessment Task 2
Term 1 2008

Name:______________________________ **Mathematics Class:**__________

Time Allowed: 60 minutes + 2 minutes reading time

Available Marks: 55

Instructions:

- Start each question on a new page.
- Attempt all six questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Questions are of approximately equal value.

Question	**1**	**2**	**3a**	**3b**	**4**	**5a**	**5b**	**6**	**Total**
PE6	/9	/8		/5	/10	/5		/11	/48
HE2			/4				/3		/7
									/55
									%

Question 1: (9 marks) **Marks**

a) Evaluate $\sum_{r=1}^{3}(-1)^{r+1}r^2$ **1**

b) What is the least integer value of n for which **3**

$$1+3+3^2+...+3^{n-1}>10^4 \ ?$$

c) (i) Find the n^{th} term of each of the series below **2**

$A_n = 3 + 6 + 12 + \ldots\ldots$
$B_n = 2 + 7 + 12 + \ldots\ldots$

(ii) Deduce from the terms of the series, A_n and B_n, the fourth term of the following series, C_n where **1**

$C_n = 10 + 26 + 48 + \ldots\ldots$

(iii) What is the n^{th} term of the series C_n? **2**

Question 2: (8 marks) Start a new page

a) For what values of x is the graph of the function $f(x)=2x^3-6x$ both concave upwards and decreasing? **2**

b) Ada borrows \$240 000 to purchase a house. A compound interest rate of 6% per annum is calculated on the balance of the loan at the end of each month. Equal monthly repayments of \$$W$ are made at the end of each month, immediately after the interest calculation. The loan is to be repaid over 20 years.

(i) Show that A_2, the amount owing on the loan after 2 months is given by **1**

$$A_2 = 240000(1\cdot005)^2 - W(1+1\cdot005)$$

(ii) Deduce a similar expression for the amount owing after 20 years **1**

(iii) Show that **2**

$$W=\frac{1200\times(1\cdot005)^{240}}{(1\cdot005)^{240}-1}$$

(iv) When does the balance owing first fall below \$200 000? Answer correct to the nearest month. **2**

Question 3: (9 marks) Start a new page **Marks**

a) Use the Mathematical induction to prove that **4**

$$\sin(x+n\pi)=(-1)^n \sin x \quad \text{for } n=1, 2, 3, \ldots..$$

b) Students at International High School must study at least one of the two languages, English and Mandarin. At a meeting of 28 students from the school, 18 study English and 22 study Mandarin.

(i) Draw a Venn (or similar) diagram illustrating this information. **2**

(ii) Hence find the probability that at the meeting:

(α) one randomly selected student studies the subject of English. **1**

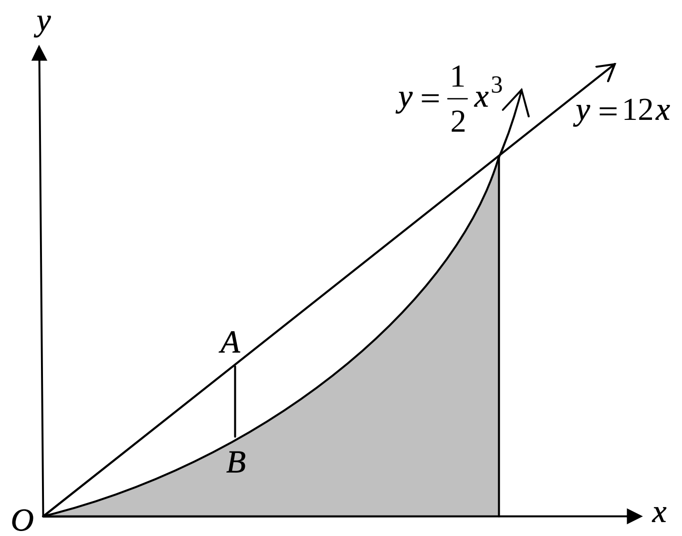
(β) two randomly selected students both study the subject of English. **1**

(γ) one randomly selected student studies both the languages specified**.** **1**

Question 4: (10 marks) **Start a new page**

a) A linear pipe is placed above a ski slope as shown below. The pipe and the cable are defined by the equations $y=12x$ and $y=\frac{1}{2}x^3$ respectively.

Vertical supports for the pipe are constructed along the slope under the pipe as illustrated in the diagram by the vertical support *AB*. Dimensions are in metres.



(i) Show that the height h of each support is given by $h=\frac{x}{2}(24-x^2)$ **2**

(ii) Find the height of the tallest vertical support that can be placed between the pipe and the ski slope within the context of the diagram. **4**

Question 4 continues

Question 4 continued: **Marks**

b) Tom contributes to a superannuation fund. At the start of every quarter (of a year), he contributes \$250. The investment pays interest at 8% per annum compounded quarterly. This contribution continues for 30 years.

(i) What amount does Tom contribute altogether? **1**

(ii) How much does the initial contribution of \$250 reach at the end of thirty years? **1**

(iii) Find the total value of Tom's fund after thirty years. **2**

Question 5: (8 marks) **Start a new page**

a) Jenny devises a game of chance for one person and plays it herself. She throws two unbiased dice repeatedly until the sum of the numbers displayed is either 9 or 12. If the sum is 9, Jenny wins the game. If the sum is 12, Jenny loses the game. If the sum is any other number, then Jenny throws again.

(i) Show that the probability that Jenny wins on the first throw is $\frac{1}{9}$. **1**

(ii) Show that the probability that a game continues with Jenny winning on the second throw is given by $\frac{1}{9}\times\frac{31}{36}$. **2**

(iii) Use your knowledge of series to find the probability that Jenny will eventually win the game. **2**

b) Use Mathematical Induction to prove that for integers $n \geq 1$, **3**

$9^{n+2} - 4^n$ is a multiple of 5.

Question 6: (11 marks) **Start a new page** **Marks**

a) A continuous function $y = f(x)$ has its second derivative defined by **2**

$$f''(x) = -\frac{4x(3 - x^2)}{(1 + x^2)^3}$$

The function $f(x)$ has three points of inflexion. Two of these points are at $(\sqrt{3}, \frac{\sqrt{3}}{2})$ and $(-\sqrt{3}, -\frac{\sqrt{3}}{2})$. Show that there is a third point of inflexion.

b) Consider the function $y = \dfrac{2x}{x^2 + 1}$.

(i) Show that this function has two stationary points. **3**

(ii) This function has its second derivative defined by **3**

$$f''(x) = -\frac{4x(3 - x^2)}{(1 + x^2)^3}$$

Hence, or otherwise, classify the stationary points found in part (i).

(iii) Using the information in both part a) and b) (i) and (ii), sketch the graph of $y = \dfrac{2x}{x^2 + 1}$ showing all of its important features including those found above. **3**

End of Paper

NSGHS TERM 1 2007
EXT 1 MATHEMATICS
TASK
SOLUTIONS

Question 1

a) $\sum_{r=1}^{3}(-1)^{r+1}r^2 = 1-4+9$
$= 6$

b)
$S_n = \frac{1(3^n-1)}{2}$

$\frac{3^n-1}{2} > 10^4$

$3^n > 2.10^4+1$

$3^n > 20001$

$n > 9$ [Trial/Error]

$n = 10$

c) (i) $A_n: T_n = 3.2^{n-1}$
$B_n: T_n = 2+(n-1)5$
$= 5n-3$

(ii) $T_4 = 82$

(iii) $T_n = 2(3.2^{n-1}+5n-3)$
$= 3.2^n+10n-6$

Question 2

a) $f'(x) = 6x^2-6$
$f''(x) = 12x$

Decreasing function:
$f'(x) < 0$
$6(x+1)(x-1) < 0$
$-1 < x < 1$ ①

Concave upward:
$f''(x) > 0$ ie $x > 0$ ②

Solving ①, ② simultaneously
$0 < x < 1$

QUESTION 2b)

2b) i) Let A_n be amount owing after n month

(i) $A_1 = 240000(1.005) - W$
$\therefore A_2 = A_1(1.005) - W$
$= [240000(1.005) - W]1.005 - W$
$= 240000(1.005)^2 - W(1+1.005)$

(ii) $A_{240} = 240000(1.005)^{240} - W(1+1.005+\dots+(1.005)^{239})$

(iii) Let $A_{240} = 0$
$\frac{W(1.((1.005)^{240}-1))}{0.005} = 240000(1.005)^{240}$
$\therefore W = \frac{1200(1.005)^{240}}{(1.005)^{240}-1}$

iv) $A < 200000$
Now
$A_n = 240000(1.005)^n - \frac{W((1.005)^n-1)}{0.005}$
We know that
$W = 1719.43$
So $240000(1.005)^n - \frac{1719((1.005)^n-1)}{0.005} < 200000$
$1200(1.005)^n - 1719(1.005)^n + 1719 < 1000$
$(1.005)^n(1200-1719) < -719$
$519(1.005)^n > 719$
$(1.005)^n > \frac{719}{519}$ (1.3
By calculator or logarithms $n > 65.3$
ie $n = 66$
66 months

OUESTION 3

a) $\sin(x+n\pi) = (-1)^n \sin x$

Step 1

Test for $n = 1$

$LHS = \sin(x+\pi)$

$RHS = -\sin x$

$LHS = \sin(x+\pi)$

$= \sin x \cos\pi + \cos x \underbrace{\sin\pi}_{0}$

$= -\sin x$

$= RHS$

Step 2

Assume that

$\sin(x+k\pi) = (-1)^k \sin x$

and hence show that

$\sin(x+(k+1)\pi) = (-1)^{k+1} \sin x$

$LHS = \sin(x+(k+1)\pi)$

$= \sin((x+k\pi)+\pi)$

$= \sin(x+k\pi)\cos\pi + \cos(x+k\pi)\sin\pi$

$= -\sin(x+k\pi)$

$= (-1)(-1)^k \sin x$ [By assumption]

$= (-1)^{k+1} \sin x$

$= RHS$

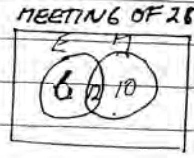
Step 3

Since the result is true for $n = 1$ and is true for $n = k+1$ if true for $n = k$ then it is true for $n = 2$ and so on for $n = 3, 4, 5, ---$

b)

MEETING OF 28

(I)



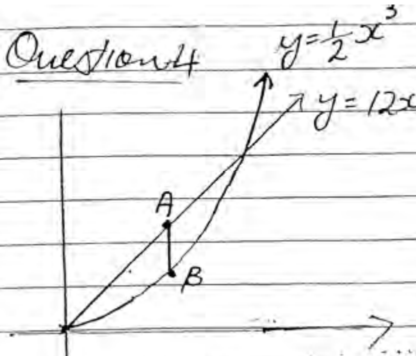
(II) (α) $\frac{9}{14}$

(β) $\frac{18}{28} \times \frac{17}{27} = \frac{17}{42}$

β $\frac{3}{7}$

Question 4

a)



Let A be (x, y) ie $(x, 12x)$
B be (x, Y) ie $(x, \frac{1}{2}x^3)$

$AB = 12x - \frac{1}{2}x^3$

(i) ie $h = \frac{1}{2}x(24 - x^2)$

Find $\frac{dh}{dx}$

(ii) $\frac{dh}{dx} = 12 - \frac{3}{2}x^2$

Put $\frac{dh}{dx} = 0$

$\frac{3x^2}{2} = 12$

$3x^2 = 24$

$x = 2\sqrt{2}$

$\therefore h = \frac{2\sqrt{2}}{2}(24 - (2\sqrt{2})^2)$

$= \sqrt{2}(24 - 8)$

$= 16\sqrt{2}$

Check Maximum

$\frac{d^2h}{dx^2} = -3x$

$= -48\sqrt{2} < 0$

implies at maximum value

Q4b)

b (i) \$30000

(ii) $250(1.02)^{120}$

\$2691.29

(iii)

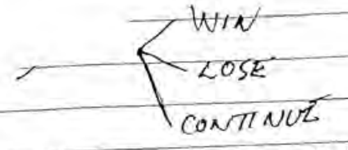
$V_{120} = 250(1.02 + (1.02)^2 + (1.02)^3 + \cdots (1.02)$

$= 250 \frac{1.02((1.02)^{120} - 1)}{0.02}$

\$124 505.83 (2)

QUESTIONS

a)



SUM OF 9 : (4,5)(5,4)(3,6)(6,3)

$P\left(\text{SUM OF } 9\right) = \frac{1}{9}$

$P\left(\text{SUM OF } 12\right) = \frac{1}{36}$

$P\left(\text{ANOTHER SUM}\right) = \frac{31}{36}$

(i) $P(W) = \frac{1}{9}$

(ii) $P\left(\text{ANOTHER SUM Then SUM OF } 9\right)$

ie $P\left(\text{ANOTHER SUM}\right) \cdot P\left(\text{SUM OF } 9\right) = \frac{31}{36} \cdot \frac{1}{9}$

(iii) $= \frac{1}{9} \cdot \frac{31}{36}$

THROW 1 ... THROW 2 ... THROW 3

$\frac{31}{36}\frac{1}{9} \quad \frac{31}{36}\frac{31}{36}\frac{1}{9} \quad \frac{31}{36}\frac{31}{36} \ldots \frac{1}{9}$

ie $S = \frac{1}{9} + \frac{1}{9}\frac{31}{36} + \frac{1}{9}\frac{31}{36}\frac{31}{36}$

$= \dfrac{\frac{1}{9}}{1 - \frac{31}{36}}$

$= \dfrac{\frac{4}{36}}{\frac{5}{36}} = \frac{4}{5}$

b) Step 1:

For $n=1$, $9^3 - 4 = 725$ which is a multiple of 5

Step 2:

Assume the formula true for $n = k$ ie $9^{k+2} - 4^k = 5N$ and show that it follows true for $n = k+1$ ie show $9^{k+3} - 4^{k+1} = 5M$ (N, M INTEGERS)

$LHS = 9^{k+3} - 4^{k+1}$

$= 9.9^{k+2} - 4.4^k$

$= 9(5N + 4^k) - 4.4^k$

$= 45N + 5.4^k$

$= 5(9N + 4^k)$

$= 5M$

$= RHS$

Step 3 Since the result is true for $n=1$ and is true for $n=k+1$ if true for $n=k$, then it is true for $n=2$ and so on for all $n = 3, 4, \ldots$

Question 6

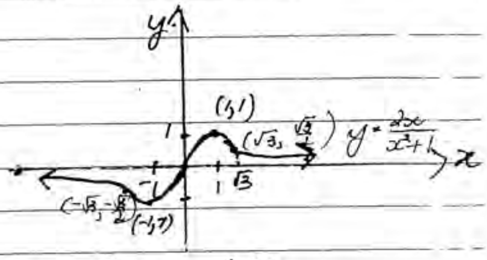
a) $f''(x) = -\dfrac{4x(3-x^2)}{(1+x^2)^3}$

$f''(0) = 0$

AND $f''(-1) = \dfrac{4(2)}{8} > 0 \Rightarrow$ CONC UP

$f''(1) = -\dfrac{4(2)}{8} < 0$ CONCAVE DOWNWARD

$(0,0)$ is a point of inflexion



Asymptote

$y = \lim_{x \to \infty} \dfrac{2x}{x^2+1}$

ie $y = 0$

b) $y = \dfrac{2x}{x^2+1}$

$y' = \dfrac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$

$= \dfrac{2-2x^2}{(x^2+1)^2}$

Put $y' = 0$ for stationary points

$\dfrac{2-2x^2}{(x^2+1)^2} = 0$

$1 - x^2 = 0$

$x = \pm 1$

$(1,1)$ and $(-1,-1)$

Test $x = 1$

$f''(1) = -\dfrac{4.2}{8} < 0 \Rightarrow$ CONCAVE DOWN

$f''(-1) = -\dfrac{-4.2}{8} > 0 \Rightarrow$ CONCAVE UPWARD

$(-1,-1)$ is a minimum turning point

$(1,1)$ is a maximum turning point

Extra Detailed Solutions

(2) (b) Let A_n be the amount owing after ***n*** months.

(i) $A_1 = 240000(1\cdot 005) - W$

$\therefore A_2 = A_1(1\cdot 005) - W$

$= \left[240000(1\cdot 005) - W\right](1\cdot 005) - W$

$= 240000(1\cdot 005)^2 - W(1 + 1\cdot 005)$

(ii) $A_{240} = 240000(1\cdot 005)^{240} - W\left(1 + 1\cdot 005 + ... + 1\cdot 005^{239}\right)$

(iii) Let $A_{240} = 0$

$\therefore 240000(1\cdot 005)^{240} - W\left(1 + 1\cdot 005 + ... + 1\cdot 005^{239}\right) = 0$

$\therefore \dfrac{W\left(1\cdot 005^{240} - 1\right)}{0\cdot 005} = 240000(1\cdot 005)^{240}$

$\therefore W = \dfrac{1200(1\cdot 005)^{240}}{1\cdot 005^{240} - 1}$

$\therefore W = 1719\cdot 43$

(iv) When is $A_n < 200\ 000$

Now $A_n = 240000(1\cdot 005)^n - \dfrac{W\left(1\cdot 005^n - 1\right)}{0\cdot 005}$

We know that $W = 1719\cdot 43$

$\therefore 240\ 000(1\cdot 005)^n - \dfrac{1719\cdot 43\left(1\cdot 005^n - 1\right)}{0\cdot 005} < 200\ 000$

$\therefore 1200(1\cdot 005)^n - 1719\cdot 43\left(1\cdot 005^n\right) + 1719\cdot 43 < 1000$

$\therefore 1\cdot 005^n(1200 - 1719\cdot 43) < -719\cdot 43$

$\therefore 519\left(1\cdot 005^n\right) > 719\cdot 43$

$\therefore 1\cdot 005^n > \dfrac{719\cdot 43}{519}$

$\therefore n > 65\cdot 3$

So 66 months

ALTERNATIVE PROBLEM for 3(a)

(a) $\cos(x+n\pi)=(-1)^n\cos x$

Test n = 1:

$$\begin{aligned}\text{LHS} &= \cos(x+\pi)\\ &= \cos(\pi+x)\\ &= -\cos x \qquad \left[\text{angles of any magnitude}, 3^{\text{rd}} \text{ quadrant}\right]\\ &= (-1)^1\cos x\\ &= \text{RHS}\end{aligned}$$

So it is true for n = 1.

Assume true for $n = k$, ie $\cos(x+k\pi)=(-1)^k\cos x$ (1)

NTP true for $n=k+1$, ie $\cos\left[x+(k+1)\pi\right]=(-1)^{k+1}\cos x$

$$\begin{aligned}\cos\left[x+(k+1)\pi\right] &= \cos\left[\pi+(x+k\pi)\right]\\ &= -\cos(x+k\pi) \qquad [\text{angles of any magnitude}]\\ &= -\left[(-1)^k\cos x\right] \qquad \left[\text{from (1)}\right]\\ &= (-1)^{k+1}\cos x\end{aligned}$$

So if $n = k$ is true then it is true for $n=k+1$ and so by the principle of mathematical induction it is true for all $n\geq 1$

(4) (b) 8% pa = 2% per quarter; 30 years = 120 quarters

(i) $250\times 120=30\,000$

(ii) The first \$250 is invested for 120 quarters and so accrues $250(1\cdot 02)^{120}$

(iii) The next \$250 is invested for 199 quarters and so accrues $250(1\cdot 02)^{119}$, and so on until the last \$250 accrues $250(1\cdot 02)$.

The total lump sum, \$$L$ is given by:

$$\begin{aligned}L &= 250(1\cdot 02)^{120}+250(1\cdot 02)^{119}+...+250(1\cdot 02)\\ &= 250\left[1\cdot 02+...+1\cdot 02^{120}\right]\\ &= 250\times S_{120} \qquad [a=1\cdot 02, r=1\cdot 02]\\ &= 250\times\frac{1\cdot 02\left(1\cdot 02^{120}-1\right)}{1\cdot 02-1}\\ &= 250\times 51\left(1\cdot 02^{120}-1\right)\\ &\approx 124\,505\cdot 83\end{aligned}$$

Tom earns \$$124\,505\cdot 83$

Question 4:

a) (i) $AB = h = 12x - \frac{1}{2}x^3$ (1) As $A = (x, 12x)$ & $B = (x, \frac{1}{2}x^3)$

$= \frac{x}{2}(24 - x^2)$ (1)

$\frac{}{2}$

(ii) $\frac{dh}{dx} = 12 - \frac{3}{2}x^2$

$\frac{d^2h}{dx^2} = -3x$ } (1)

at max/min, $\frac{dh}{dx} = 0 \therefore 12 - \frac{3x^2}{2} = 0$

$\frac{3x^2}{2} = 12$

$x^2 = 8$ (1)

$x = \sqrt{8}$ as $x > 0$ ($\frac{1}{2}$)

$\frac{}{4}$

then $\frac{d^2h}{dx^2} = -3(\sqrt{8}) < 0$ ⇒ a maximum occurs ($\frac{1}{2}$)

$\therefore$ max height $= 12\sqrt{8} - \frac{1}{2}(\sqrt{8})^3$ m

$= 12\sqrt{8} - 4\sqrt{8}$ m

$= 16\sqrt{2}$ m (1)

$\frac{}{1}$ (b) (i) Total contributions $= 4 \times \$250 \times 30$

$= \$30000$ (1)

$\frac{}{1}$

(ii) Initial contribution grows to $\$(250)(1.02)^{120}$ (1)

$= \$2691.29$

(iii) second contribution grows to $\$(250)(1.02)^{119}$

Third contribution grows to $\$250\,(1.02)^{118}$ } ($\frac{1}{2}$)

$\therefore$ Total $= 250(1.02) + 250(1.02)^2 + 250(1.02)^3 + \ldots + 250(1.02)^{120}$

$= \frac{a(r^n - 1)}{r - 1}$ where $a = 250(1.02)$ ($\frac{1}{2}$)

$r = 1.02$

$\frac{}{2}$

$n = 120$

$= \frac{250(1.02)(1.02^{120} - 1)}{0.02}$ ($\frac{1}{2}$)

$= 124505.828\ldots$

($\frac{1}{2}$)

$\therefore$ Fund is worth $\$124505.83$ (nearest c)